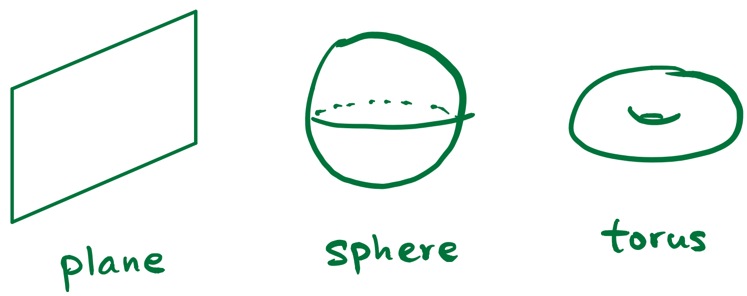
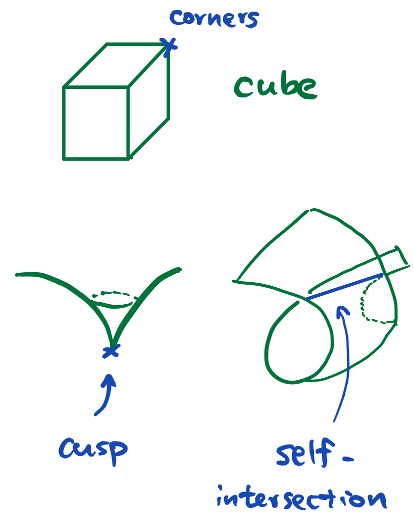
§ Surfaces in $\mathbb{R}^3$

Q: Which one do we want to consider as "surfaces"?



YES



NO

Basic idea: A "surface" is an object that ① locally ② looks like a piece of $\mathbb{R}^2$.

Definition: A (regular) surface is a subset

$$S \subseteq \mathbb{R}^3$$

s.t. $\forall\, p \in S$, $\exists$ nbd. of p (in S) $V \subseteq S$ and a smooth map (called parametrization / chart)

$$\mathbb{X} : \underset{\text{open}}{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow V$$

s.t. (1) $\mathbb{X} : \mathcal{U} \to V$ is a homeomorphism.

* (2) The differential $d\mathbb{X}|_q$ is 1-1 $\forall\, q \in \mathcal{U}$.

Explanation on *: More explicitly,

$$X(u,v) = \big(\underbrace{x(u,v),\ y(u,v),\ z(u,v)}_{\text{smooth functions}} \big), \quad q = (u,v) \in \mathcal{U}$$

The differential of X at $q \in \mathcal{U}$ is a linear map

$$dX\big|_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

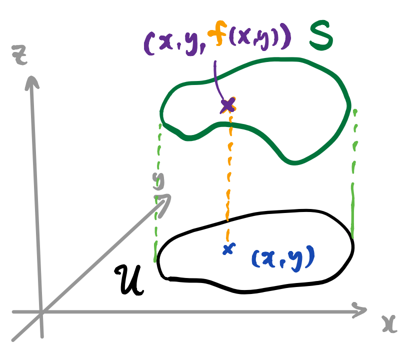
which can be expressed in matrix form (w.r.t. std basis)

$$dX\big|_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}\Bigg|_q =: \begin{pmatrix} | & | \\ \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ | & | \end{pmatrix}\Bigg|_q$$

$$\boxed{dX\big|_q \text{ is } 1\text{-}1 \iff \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v} \text{ are linearly independent.}}$$

Example 1: Graphical surfaces

Given a smooth function $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$



$$S = \text{graph}(f)$$
$$= \{ z = f(x,y) \mid (x,y) \in \mathcal{U} \}$$

is a (regular) surface.

Why? Consider the smooth map

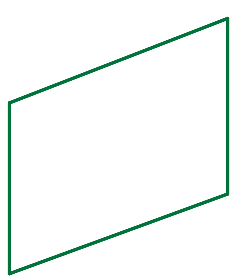
$$\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S \subseteq \mathbb{R}^3$$
$$(u,v) \longmapsto (u, v, f(u,v))$$

Clearly, $\mathbf{X} : \mathcal{U} \to S$ is a homeomorphism.

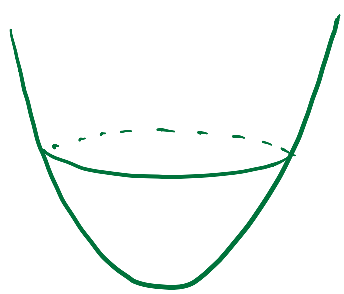
$$\left.\begin{aligned} \frac{\partial \mathbf{X}}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right) \\ \frac{\partial \mathbf{X}}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right) \end{aligned}\right\} \text{ always linearly independent.}$$

Examples of graphical surfaces

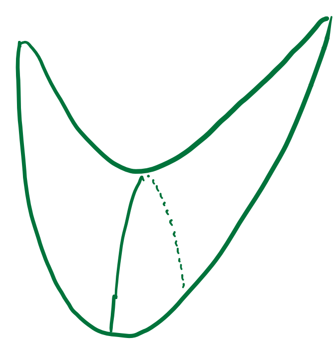
- $f(x,y) = ax + by + c \implies$ planes
- $f(x,y) = x^2 + y^2 \implies$ paraboloid
- $f(x,y) = x^2 - y^2 \implies$ hyperboloid



plane



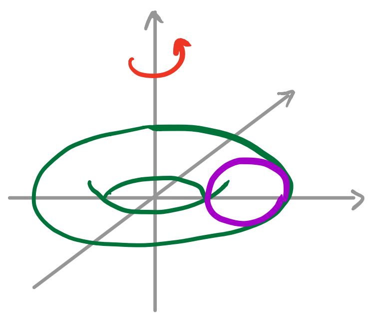
paraboloid



hyperboloid

Note: The entire surface can be covered by 1 chart.

Example 2 : Torus of revolution



$$S = \{ (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2 \}$$

where $a > r > 0$ are constants.

is a surface.

Exercise: Prove this.

How many parametrizations do we need to cover the whole torus?